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On the perturbation of the observability equation in linear control systems

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Abstract The aim of this paper is to investigate stability and sensitivity of the observability variable in linear control systems, (LCS) for short. We first present two results of Hölder continuity in the abstract framework of the ordinary differential equation initial-value problem $x'(t) = f(t, x(t)), x(t_0) = x_0$. Afterwards, we apply our results to automatic systems, providing henceforth the sharpest bounds for the parametric input-output relation in LCS.

Keywords Linear control systems · Ordinary differential equations · Perturbation · Observability · Quantitative stability

1 Introduction

The classical Cauchy-Lipschitz problem of ordinary differential equation (ODE) initial-value problem $x'(t) = f(t, x(t)), x(t_0) = x_0$ is considered. One of the fashion topics in this problem concerns sensitivity of the solutions. In regard to perturbation on the initial value, results are now available. In this respect we refer the reader to the recent paper by Jong-Shi Pang and David Stewart [4] and references therein. In some systems, some external parameters may perturb their states. A typical instance in this context we can give is the input-output relation in linear control where all of the data in both evolution and observability equations are permanently subject to non-negligible change. In this case, one can possibly convert the question to one of change of initial conditions, by augmenting the given state variables of the ODE, which may work pretty well and fit many sensitivity problems to a nice abstract unifying framework. However, we believe that, for heterogeneity considerations, the applicability of these eventual results-that can be obtained via change state variable approach-may cause some hard time for some engineers and people working in automatic at the practical level. They actually need to separate parameters and compute the sharpest error bound for each parameter to obtain a better performance of their systems. The present note poses new

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questions in this direction on parametric ordinary differential equations and examine the feasibility of further treatments. In particular, can we measure the distance between solutions to nominal and perturbed problems of ODE?

2 A Lipschitz estimate

Let I be the interval of interest containing 0 (a translation allows us to recapture the case of $t_0 > 0$). Of course, $I \subseteq R$. Let V a neighborhood of R^m and $f\colon I \times V \to R^m$ a given function.

$$S(f, x_0) \begin{cases} x'(t) = f(t, x(t)) \\ x(0) = x_0. \end{cases} \tag{1}$$

The norm of R^m is denoted by $|.|$. To introduce the perturbed system of $S(f, x_0)$, we first speak about an external parameter λ who comes from another space. Formally, λ usually belongs to some open in some Euclidian space, say $\lambda \in U \subset R^n$. The norm of R^n is indifferently denoted by $\|.\|$. Thus, consider the perturbed form of f as follows $f\colon I \times V \times U \to R^m$. The perturbed system is then the following: For each $\lambda \in U$ and an initial condition $x_\lambda(0) = x_{\lambda_0}$, one should seek for $x(\lambda) := x_\lambda\colon I \to V$ continuous and differentiable on the interior of I such that

$$S(f_\lambda, x_{\lambda_0}) = \begin{cases} x'_\lambda(t) = f(t, x_\lambda(t), \lambda) \\ x_\lambda(0) = x_{\lambda_0}. \end{cases} \tag{2}$$

To further simplify the notations we adopt $f(.,.,\lambda) = f_\lambda\colon I \times V \to R^m$. The perturbed system $S_\lambda(f, x_{\lambda_0})$ then becomes

$$S_\lambda \begin{cases} x'_\lambda(t) = f_\lambda(t, x_\lambda(t)) \\ x_\lambda(0) = x_{\lambda_0}. \end{cases} \tag{3}$$

The nominal value of the parameter λ is denoted by $\bar{\lambda}$. In this respect, we consider that $f_{\bar{\lambda}} = f$, $x_{\bar{\lambda}} = x$ and $x_{\bar{\lambda}}(0) = x_0$, i.e., $S_{\bar{\lambda}} = S(f, x_0)$.

We shall also introduce the sequential perturbed problem of ODE initial-value problem:

$$S_n := S_n(f_n, x_0^n) = \begin{cases} x'_n(t) = f_n(t, x_n(t)) \\ x_n(0) = x_0^n, \end{cases} \tag{4}$$

where f_n is a sequence of functions converging to f in some sense and x_0^n is a given initial value of the perturbed problem. We need, for $t \in [0, T]$, to introduce the following distance

$$D_{r(t)}(f, f_n) = \max_{x \in B(0, r(t))} |f_n(t, x) - f(t, x)|$$

and suppose that $x_n(t) \in B(0, r(t))$ for an increasing map $r\colon [0, T] \to R$ such that $r(t) > 0$ for all $t > 0$.

We now claim our first result which is in concern with the sequential perturbation.

Theorem 1 *Assume that f is L-Lipschitz in x uniformly in t. Then,*

$$|x(t) - x_n(t)| \leq e^{Lt}|x_0 - x_0^n| + \frac{L'}{L}(e^{Lt} - 1)D_{r(T)}(f, f_n). \tag{5}$$

Proof Write $|x(t) - x_n(t)|^2 = \langle x(t) - x_n(t), x(t) - x_n(t)\rangle$ and see that

$$\frac{d}{dt}|x(t) - x_n(t)|^2 = 2\langle x(t) - x_n(t), x'(t) - x_n'(t)\rangle \tag{6}$$

$$= 2\langle x(t) - x_n(t), f(t, x(t)) - f_n(t, x_n(t))\rangle. \tag{7}$$

Thus, Cauchy-Shwarz and triangular inequalities imply

$$\begin{aligned}\frac{d}{dt}|x(t) - x_n(t)|^2 &\le 2|x(t) - x_n(t)||f(t, x(t)) - f_n(t, x_n(t))|\\ &\le 2|x(t) - x_n(t)|[|f(t, x(t)) - f(t, x_n(t))|\\ &\quad +|f(t, x_n(t)) - f_n(t, x_n(t))|].\end{aligned}$$

Whence, the Lipschitz property of f leads to

$$\frac{d}{dt}|x(t) - x_n(t)|^2 \le 2L|x(t) - x_n(t)|^2 + 2|x(t) - x_n(t)||f(t, x_n(t)) - f_n(t, x_n(t))|.$$

Then, the Gronwall inequality allows us to conclude

$$|x(t) - x_n(t)| \le e^{Lt}|x_0 - x_0^n| + \frac{L'}{L}(e^{Lt} - 1)D_{r(T)}(f, f_n), \tag{8}$$

completing the proof.

Theorem 2 *Assume for some* $L, L' > 0$ *that*

(h_1) *f is L-Lipschitz continuous w. r to x, uniformly in t and* λ*;*
(h_2) *f is* L'*-Lipschitz (resp.* (c, α)*-Hölder) continuous w. r to* λ*, uniformly in t and x.*

Then, for all $t \in [0, T]$*, the following estimate holds*

$$|x(t) - x_\lambda(t)| \le e^{Lt}|x_0 - x_{\lambda_0}| + \frac{L}{L'}(e^{Lt} - 1)\|\lambda - \bar{\lambda}\| \tag{9}$$

$$(resp.\ |x(t) - x_\lambda(t)| \le e^{Lt}|x_0 - x_{\lambda_0}| + \frac{c}{L}(e^{Lt} - 1)\|\lambda - \bar{\lambda}\|^\alpha). \tag{10}$$

Proof The proof is based on similar techniques to those of Theorem 1 which leads to the required estimate without too much efforts.

Remark 1 One quick remark is that $x(t)$ (resp. $x_\lambda(t)$) is a solution to $S(f, x_0)$ (resp. S_λ) if, and only if

$$x(t) = x_0 + \int_0^t f(s, x(s))ds\ \ (resp.\ x_\lambda(t) = x_{\lambda_0} + \int_0^t f_\lambda(s, x_\lambda(s))ds), \quad \forall t \in [0, T], \tag{11}$$

which leads, under assumptions $h_1) - h_2)$, to the following estimate of the integral parametric solution:

$$\int_0^t |x(s) - x_\lambda(s)|ds \le \frac{(e^{Lt} - 1)}{L}(|x_0 - x_{\lambda_0}| + L'T\|\lambda - \bar{\lambda}\|) \quad \forall t \in [0, T]. \tag{12}$$

Indeed, by (11), we can write

$$x(t) - x_\lambda(t) = x_0 - x_{\lambda 0} + \int_0^t [f(s, x(s)) - f_\lambda(s, x_\lambda(s))]ds$$

$$= x_0 - x_{\lambda 0} + \int_0^t [f(s, x(s)) - f_\lambda(s, x(s))]ds$$

$$+ \int_0^t [f_\lambda(s, x(s)) - f_\lambda(s, x_\lambda(s))]ds.$$

Involving now $h_1) - h_2)$ and see that

$$|x(t) - x_\lambda(t)| \le |x_0 - x_{\lambda 0}| + L'T\|\lambda - \bar{\lambda}\| + L\int_0^t |x(s) - x_\lambda(s)|ds. \tag{13}$$

Therefore put $z(t) = \int_0^t |x(s) - x_\lambda(s)|ds$, $a = |x_0 - x_{\lambda 0}| + L'T\|\lambda - \bar{\lambda}\|$ and simply write

$$z'(t) \le a + Lz(t). \tag{14}$$

Then, Gronwall inequality implies that

$$z(t) + \frac{a}{L} \le (z(0) + \frac{a}{L})e^{Lt}, \;\; \forall t \in [0, T], \tag{15}$$

concluding the required estimate (12).

3 Special case: evolution and observability equations in linear automatic systems

For given $m \times m$-matrices A, B, C, D and an initial condition x_0, we consider the following standard automatic system:

$$S(A, B, u, C, D, x_0) : \begin{cases} x'(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \\ x(0) = x_0, y_0 = Cx_0 + Du(0). \end{cases} \tag{16}$$

Here, $x(t)$ is the state variable, $y(t)$ is the observability variable while $u(t)$ denotes the control of the system.

Set

$$f(t, x(t)) = Ax(t) + Bu(t). \tag{17}$$

Observe that for $t \in [0, T]$ and $x, \tilde{x} \in R^m$ one has $f(t, x) - f(t, \tilde{x}) = A(x - \tilde{x})$. Hence,

$$|f(t, x) - f(t, \tilde{x})| = |A(x - \tilde{x})| \le \|A\||x - \tilde{x}|. \tag{18}$$

The Lipschitz rank of f in this case is nothing else but the norm of the matrix A, i.e., $L = \|A\|$. In some typical situations, it suffices to look for eigenvalues of A to compute L.

Perturbation of A Now, we let the initial condition to stay fixed and propose, as a first step, a particular perturbation (related to a parameter λ) which will concern some practical examples we want to point out later. Without loss of generality take $m = 2$, $n = 2$. The initial value of λ is denoted by $\bar{\lambda}$. In some application we will talk about the ideal instance of $\bar{\lambda}$ where the influence of the parameter λ is negligible at some time. Consider a matrix $A = (a_{ij})_{i,j=1,2}$ and its prototype perturbed form A_λ defined as follows $a^\lambda_{11} = a_{11} + \lambda_1$, $a^\lambda_{22} = a_{22} + \lambda_2$, $a^\lambda_{21} = a_{21}$ and $a^\lambda_{12} = a_{12}$. Take now a sufficiently large real number $r(\lambda) > 0$ depending on λ such that $|x_\lambda|, |x| \leq r(\lambda)$. Assume that for some $r > 0$, $r(\lambda) \leq r$ for all $\lambda \in \mathcal{V}(\bar{\lambda})$. Then, we see that

$$\|A_\lambda x - Ax\| \leq r\|\lambda\|.$$

Define therefore,

$$f_\lambda(t, x(t)) = A_\lambda x(t) + Bu(t). \tag{19}$$

Using the estimate (9) (with $L' = r$) of Theorem 2, we deduce

$$|x(t) - x_\lambda(t)| \leq \delta(A, r, t)\|\lambda\|, \tag{20}$$

where

$$\delta(A, r, t) = \frac{r}{\|A\|}(e^{\|A\|t} - 1).$$

We are presently in a position to claim a sensitivity result for the observability of the system $S(\lambda, A, B, u, C, D, x_{\lambda_0}) := S_\lambda(A, B, u, C, D, x_{\lambda_0})$ for which we consider that $\bar{\lambda} = 0$.

Theorem 3 *For any λ around $\bar{\lambda} = 0$ and all $t \in [0, T]$ one has*

$$|y(t) - y_\lambda(t)| \leq \delta(A, r, t)\|C\|\|\lambda\|. \tag{21}$$

Corollary 1 *If the initial condition is also subject to change we obtain the following estimate.*

$$|y(t) - y_\lambda(t)| \leq e^{Lt}\|C\||x_0 - x_{\lambda_0}| + \|C\|\delta(A, r, t)\|\lambda\|. \tag{22}$$

Physical interpretation The estimate (22) shows that the difference between values taken respectively by nominal exit $(y(t))$ and perturbed exit (y_λ) does not increase faster than the terms $\|C\|e^{Lt}$ and $\delta(A, r, t)\|C\|$. In other words, our system is not sensitive to the perturbation of the parameter λ and that of the initial condition x_0, in the sense that a small change of λ or x_0 induces only a small perturbation on the observability data, thus we can actually speak about a nice perturbation. In a forthcoming work ([1]) we develop this approach to contribute to a better understanding of perturbation phenomena arising in electrical networks together with their simulations based on toolbox Simulink.

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